Determination of the Capacitance, Inductance, and Characteristic Impedance of Rectangular Lines*

TSUNG-SHAN CHEN†, MEMBER, IRE

Summary—This paper determines the capacitance, inductance, and characteristic impedance of rectangular lines by the method of conformal transformation. In practical applications, such lines may be used as transmission links of RF energy, as impedance-transforming sections, or as components in electron tubes.

Formulas are given for the calculation of the parameters of rectangular lines having the following characteristics: 1) The inner conductor may have varying thickness compared with the depth of the outer conductor. 2) The axes of the conductors may coincide or may be displaced with respect to each other. 3) The edges of the inner conductor may be rounded to lessen the electrical stress occurring at sharp corners.

Excellent agreement has been obtained between the calculated results and those found by use of the relaxation method, by direct measurement of models, and by electrolytic tank measurement.

I. Introduction

The rectangular line consists of a rectangular inner conductor located symmetrically or asymmetrically inside a rectangular hollow outer conductor in a manner similar to a coaxial line. When the depth of each conductor is equal to its width, the line becomes a square line. The electric and magnetic fields in such a geometry bear a close resemblance to those in a coaxial line, especially for the case of a small inner conductor. If the ratio of width to depth in both conductors is large and the inner conductor forms a flat strip, the field patterns in the rectangular line approach those existing in a shielded strip line.[1] Thus, the rectangular line, besides being used to transmit RF energy, may serve as an impedance transformer interposed between coaxial and strip transmission lines.

This article concerns the determination of the capacitance, inductance and characteristic impedance of rectangular lines where the inner conductor may be thick or thin in comparison with the depth of the outer conductor. If the spacings between the conductors are small relative to their width and depth, the line parameters can be determined analytically even when the inner conductor is placed asymmetrically with respect to the outer one. For the purpose of reducing the electrical stress in the annular region between conductors, the edges of the inner conductor are rounded, and the effect of rounding the corners on the line parameters is evaluated.

The parameters of rectangular lines calculated by means of the formulas derived here agree very closely with the results obtained by the relaxation method, by direct measurement of full-sized models and by electrolytic tank measurement.

II. Relation Between the Inter-Conductor Capacitance and the Characteristic Impedance

The rectangular line is essentially a two-wire transmission system along which TEM waves are propagating. The velocity of propagation, when dissipation is neglected, equals the velocity of light, and is given by

$$v = 1/(\mu\epsilon)^{1/2} = 1/(LC)^{1/2}. \tag{1}$$

The characteristic impedance of such a lossless line is given by

$$Z = (L/C)^{1/2} = 1/(vC). \tag{2}$$

In MKS units, the quantities used in (1) and (2) are as follows:

Z = characteristic impedance of the line in ohms
L = inductance of the line in henries per meter
C = capacitance of the line in farads per meter
v = velocity of propagation in free space in meters per second
$= 2.998 \times 10^8$ meters per second
μ = permeability of free space
$= 4\pi \times 10^{-7} = 1.257 \times 10^{-6}$ henry per meter
ϵ = permittivity of free space
$= 1/(36\pi) \times 10^{-9} = 8.854 \times 10^{-12}$ farad per meter.

Eq. (2) shows that the evaluation of the characteristic impedance of rectangular lines reduces to the determination of the interconductor capacitance by experimental or analytical means. The experimental determination of the capacitance between the conductors can be accomplished by direct measurement of a full-scale model or by mapping the equipotentials and flux lines existing between appropriate electrode shapes placed in an electrolytic tank.

The analytical process of obtaining the capacitance is based upon the solution of Làplace's equation for a static field in two dimensions subjected to proper boundary conditions. The solution involves the determination of the potential functions, the flux lines, and the charge distribution on the electrodes. A numerical result can be found for a specific problem by use of the

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† Electron Tube Div., Radio Corp. of America, Harrison, N. J.
[1] R. M. Barrett, "Microwave printed circuits—a historical survey," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 1–9; March, 1955.

relaxation method,[2] or analytical expressions may be obtained from formal mathematics.

If the conductor geometries are simple, such as concentric spheres or parallel cylinders, Laplace's equation can be integrated formally, and boundary conditions are applied to get explicit solutions. In other cases, the two-dimensional differential equation can be solved by means of conjugate functions, the real or imaginary parts of which represent the potential or flux functions. When the conductor boundary assumes a polygon, the determination of the proper conjugated function can be effected by means of the Schwarz-Christoffel transformation.

III. Rectangular Line with Small Spacings Between Conductors

Fig. 1 illustrates the configuration of a symmetrical rectangular line. The exact determination of the capacitance by the method of conformal transformation involves hypergeometric functions and four variable parameters; the process of obtaining numerical results would be so laborious that it has not been attempted. If the sides of the conductors are large compared with their spacings, the distorted fields at the two ends along the same side of the inner conductor do not interact, and only one quarter of the cross section needs to be transformed. The interconductor capacitance can then be calculated as a combination of parallel-plate condensers formed by the walls of the conductors, plus excess capacitance caused by the disturbances of flux lines close to the corners. In fact, this method is valid whenever the short side of the inner conductor exceeds the spacing distances, as evidenced by the negligible amount of flux distortion at points not far away from the bend shown in Fig. 2.

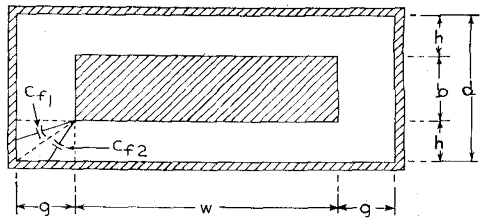

Fig. 1—The symmetrical rectangular line.

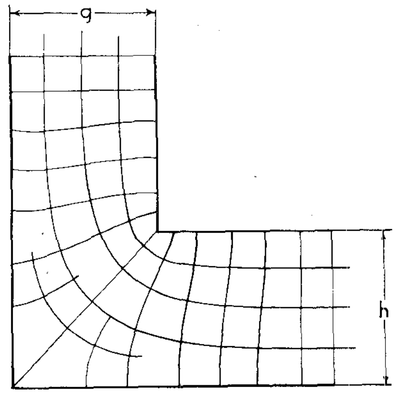

Fig. 2—Distortion of electric field at one corner of the rectangular line.

A. Line Capacitance

One corner of the line cross section assumes the shape of a right-angle bend; two successive transformations are necessary in this case as discussed in Appendix I.[3,4] The first process transforms the z-plane polygon into the real axis of the t plane, and another transformation from the w plane to the t plane relates the potentials of the two conductors to values of t. The capacitance between the conductors is evaluated by letting z as well as t take critical values which depend on the particular problem.

In the L-shaped bend, the excess or fringing capacitance caused by the disturbance of flux lines emanating at the vertical side is expressed by

$$C_{f1} = \frac{\epsilon}{\pi}\left[\log\frac{g^2+h^2}{4h^2} + 2\left(\frac{h}{g}\right)\arctan\frac{g}{h}\right]$$

farads per meter, (3)

where g is the lateral spacing and h the vertical spacing. Similarly, the equation for fringing capacitance produced by flux disturbance along half of the horizontal side is

$$C_{f2} = \frac{\epsilon}{\pi}\left[\log\frac{g^2+h^2}{4g^2} + 2\left(\frac{g}{h}\right)\arctan\frac{h}{g}\right]$$

farads per meter. (4)

On the supposition that the conductor sides are large, the fringing capacitance depends only on the spacings and not on the conductor dimensions. The ratios C_{f1}/ϵ and C_{f2}/ϵ are plotted in Fig. 3 as a function of h/g or of g/h. The total capacitance between the conductors is

$$C = 2\epsilon\left(\frac{w}{h} + \frac{b}{g}\right) + \frac{4\epsilon}{\pi}\left[\log\frac{g^2+h^2}{4h^2} + 2\left(\frac{h}{g}\right)\arctan\frac{g}{h}\right] + \frac{4\epsilon}{\pi}\left[\log\frac{g^2+h^2}{4g^2} + 2\left(\frac{g}{h}\right)\arctan\frac{h}{g}\right]$$

farads per meter, (5)

[2] R. V. Southwell, "Relaxation Methods in Engineering Science," Oxford University Press, Oxford, Eng.; 1940. "Relaxation Methods in Theoretical Physics," Oxford University Press, Oxford, Eng., vol. I, 1946; vol. II, 1956.

[3] J. J. Thomson, "Recent Researches in Electricity and Magnetism," Oxford University Press, Oxford, Eng.; 1893.

[4] J. H. Jeans, "The Mathematical Theory of Electricity and Magnetism," Cambridge University Press, Cambridge, Eng., 5th ed.; 1925.

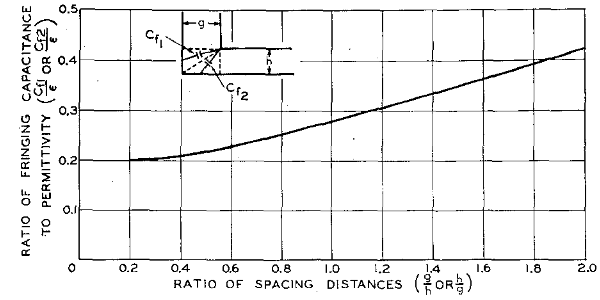

Fig. 3—Ratio of fringing capacitance to permittivity (C_{f1}/ϵ, or C_{f2}/ϵ) as a function of the spacing ratio (g/h or h/g).

where w and b are, respectively, the width and thickness of the inner conductor. If $g=h$, both (3) and (4) reduce to

$$C_f = \frac{\epsilon}{\pi}\left[\frac{\pi}{2} - \log 2\right] = 0.279\epsilon, \qquad (6)$$

and the line capacitance is expressed by

$$C = \frac{2\epsilon(w+b)}{g} + 2.232\epsilon. \qquad (7)$$

B. Line Inductance

The method of conformal transformation demonstrates that the fringing effect caused by charge concentration close to the edges of the inner conductor may be accounted for by the addition of correction lengths to the conductor sides. In Fig. 2, half of the vertical side of inner conductor should be increased by the amount

$$X_1 = \frac{1}{\pi}\left[g \log \frac{g^2+h^2}{4h^2} + 2h \arctan \frac{g}{h}\right] = g\,\frac{C_{f1}}{\epsilon}\cdot \qquad (8)$$

The extension in half of the horizontal side, $X_2 = hC_{f2}/\epsilon$, can be obtained from (8) by interchanging g and h. When the effective lengths of the sides are used in the formula for calculating the inductance of parallel-plate transmission lines, the inductance of the rectangular line L is given by

$$L = \frac{L_v L_H}{L_v + L_H} \text{ henries per meter.} \qquad (9)$$

In this expression, L_v and L_H are the inductances corresponding to the vertical and horizontal parallel-plate systems and are, respectively, given by

$$L_v = \frac{1}{2}\,\frac{\mu g}{b + \dfrac{2}{\pi}\left[g \log \dfrac{g^2+h^2}{4h^2} + 2h \arctan \dfrac{g}{h}\right]}, \qquad (10)$$

and

$$L_H = \frac{1}{2}\,\frac{\mu h}{w + \dfrac{2}{\pi}\left[h \log \dfrac{g^2+h^2}{4g^2} + 2g \arctan \dfrac{h}{g}\right]}\cdot \qquad (11)$$

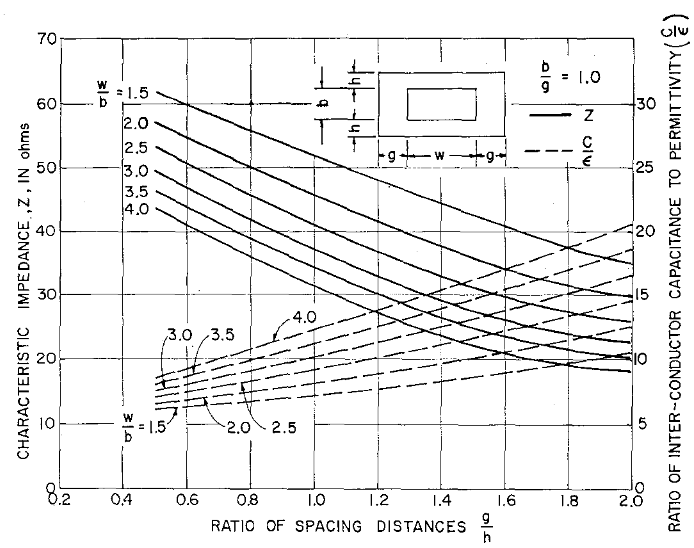

Fig. 4—Characteristic impedance and interconductor capacitance of symmetrical rectangular lines ($b/g=1$).

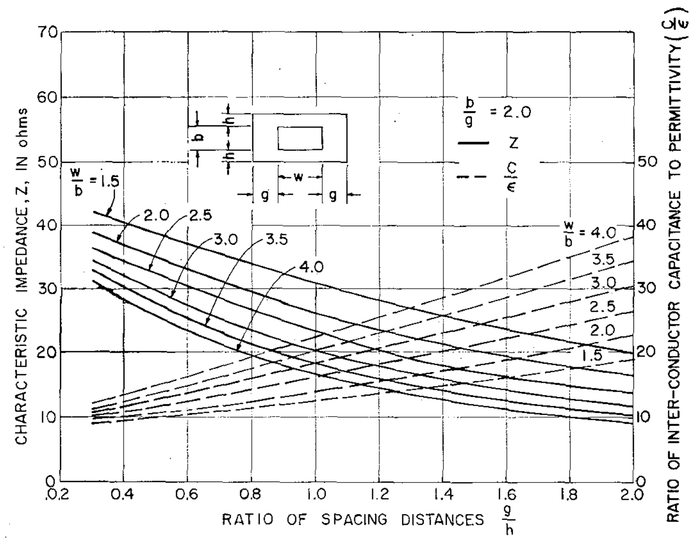

Fig. 5—Characteristic impedance and interconductor capacitance of symmetrical rectangular lines ($b/g=2$).

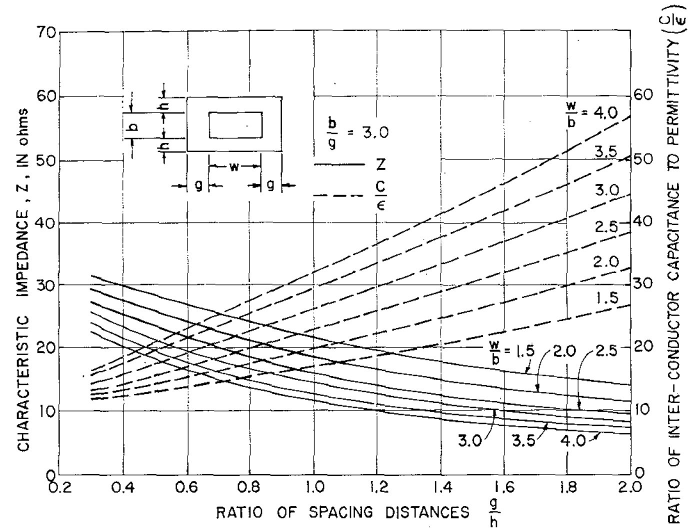

Fig. 6—Characteristic impedance and interconductor capacitance of symmetrical rectangular lines ($b/g=3$).

C. Characteristic Impedance of a Rectangular Line

When the line is concentric as shown in Fig. 1, the characteristic impedance can be obtained by use of (2), in which the capacitance given by (5) has been employed. Then

$$Z = \frac{376.62}{2\left(\frac{b}{g} + \frac{w}{h}\right) + 4\left(\frac{C_{f1}}{\epsilon} + \frac{C_{f2}}{\epsilon}\right)} \text{ ohms.} \quad (12)$$

The rectangular line in which $g=h$ has the characteristic impedance is

$$Z = \frac{376.62}{2\left(\frac{b+w}{g}\right) + 2.232} \text{ ohms.} \quad (13)$$

For a line having the dimensions, $w=0.218$, $b=0.050$, and $g=h=0.050$ inch, the line capacitance as determined from (7) is $C = (10.72+2.232)\epsilon = 12.952\epsilon = 114.677 \times 10^{-12}$ farad per meter. The inductance of the line is equivalent to that of a parallel-plate system which has a separation of 0.050 inch and an effective width of

$$2(0.218 + 0.050) + 8\left(\frac{1}{\pi}\right)\left(\frac{\pi}{2} - \log 2\right)(0.05) = 0.6438 \text{ inch.}$$

The value of this inductance is as follows:

$$L = \frac{\mu(0.05)}{0.6438} = \frac{\mu}{12.876} = 9.755 \times 10^{-8} \text{ henry per meter.}$$

The velocity of wave propagation is $v=1/(LC)^{1/2} = 2.992 \times 10^8$ meters per second, which is very close to the accepted value of 2.998×10^8. The characteristic impedance of the line is found to be 29.12 ohms.

By using relaxation calculation, W. N. Parker, of the RCA Electron Tube Division, Lancaster, Pa., has found the characteristic impedance of the same structure to be 27.8 ohms, while the result of electrolytic tank measurement conducted under his direction is slightly less than 27 ohms. The characteristic impedance and interconductor capacitance of rectangular lines have been calculated by means of (5) and (12) for a variety of dimensions; the results are plotted in Figs. 4–6.

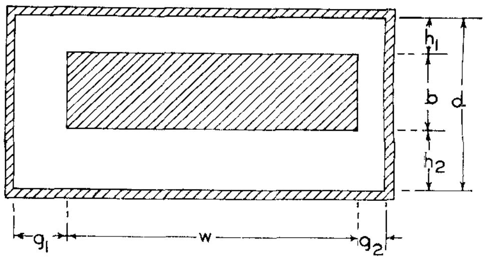

Fig. 7—Doubly eccentric rectangular line.

D. Doubly Eccentric Rectangular Line

The inner conductor may be displaced both horizontally and vertically with respect to the outer conductor so that the line becomes doubly eccentric, as shown in Fig. 7. The capacitance between the conductors becomes

$$C = \epsilon\left(\frac{b}{g_1} + \frac{b}{g_2} + \frac{w}{h_1} + \frac{w}{h_2}\right) + \sum_{m=1,2}^{n=1,2} \frac{\epsilon}{\pi}\left[\log\frac{g_n^2 + h_m^2}{4h_m^2} + 2\left(\frac{h_m}{g_n}\right)\text{arc tan}\frac{g_n}{h_m}\right] + \sum_{m=1,2}^{n=1,2} \frac{\epsilon}{\pi}\left[\log\frac{g_n^2 + h_m^2}{4g_n^2} + 2\left(\frac{g_n}{h_m}\right)\text{arc tan}\frac{h_m}{g_n}\right]. \quad (14)$$

Eight different terms of fringing capacitance are involved in the last expression. There are also eight different correction lengths for the determination of the line inductance. The characteristic impedance can be written as

$$Z = \frac{376.62}{\sum_{n=1,2} \frac{b}{g_n} + \sum_{m=1,2} \frac{w}{h_m} + \frac{1}{\epsilon}\left[\sum_{m=1,2}^{n=1,2} C_{f1}(g_n, h_m) + \sum_{m=1,2}^{n=1,2} C_{f2}(g_n, h_m)\right]}. \quad (15)$$

The capacitance of a singly eccentric line having the dimensions $w=0.120$, $b=0.041$, $g=0.027$, $h_1=0.019$ and $h_2=0.0462$ inch as calculated from (14) is

$$C = [6.312+2.603+3.034+2(0.244+0.342+0.224+0.382)]\epsilon = 14.336\epsilon = 126.931 \times 10^{-12} \text{ farad per meter.}$$

After the computation of the correction lengths from equations similar to (8), the component inductances pertaining to the upper, lower, right and left parallel-plate guides are

$$L_{\text{upper}} = \frac{\mu(0.019)}{0.120 + 2(0.0046)} = 0.1466\mu,$$

$$L_{\text{lower}} = \frac{\mu(0.0462)}{0.120 + 2(0.0176)} = 0.2975\mu,$$

$$L_{\text{right}} = L_{\text{left}} = \frac{\mu(0.027)}{0.041 + 0.0092 + 0.0061} = 0.479\mu.$$

The sum of the reciprocals of the component inductances equals the reciprocal of the line inductance. This relation yields the value

$$L = \mu/14.357 = 8.756 \times 10^{-8} \text{ henry per meter.}$$

The velocity of propagation as obtained from $v=1/(LC)^{1/2}$ is 3×10^{-8} meters per second, and the characteristic impedance from $Z=1/vC$ is 26.2 ohms. The capacitance of a full-sized model of such a line was measured by R. Schumacher of RCA Electron Tube Division, Lancaster, Pa., and found to be 128.9×10^{-12} farad per meter. This value of capacitance gives an impedance of 25.9 ohms.

The preceding formulas for calculation of the parameters of rectangular lines give accurate results when g and h are, respectively, less than w and b. This conclusion is substantiated by the flux mapping of the L-shaped bend and by comparison of the result so calculated with the exact solution. The flux distribution becomes almost uniform at a distance from the corner equal to half the spacing. If the thickness of the inner conductor is too small, the distorted fields at the two ends of the short side interfere, and the formulas so far derived do not provide sufficient accuracy.

IV. Rectangular Lines with Thin Inner Conductors

Rectangular lines often occur where one side of the inner conductor is smaller than the spacing between the two conductors. The line capacitance in such cases has been treated by J. D. Cockcroft[5] as discussed in Appendix II, in which the Schwarz-Christoffel transformation is applied to one half of the conductor cross section. The exact solution involves elliptic functions and theta functions, and the determination of the variables to give required conductor dimensions becomes very complex. For a certain range of conductor spacings and sizes, the expressions take comparatively simple form. The fictitious increase in the conductor side resulting from charge concentration at one complete corner is given by

$$X = \frac{1}{2k} \log \frac{2(k)^{1/2}}{q^{1/2}(1-k)} - \left(K - \frac{K'}{2}\right)\left(\frac{1+k}{2k} - \frac{\pi}{2kK}\right), \tag{16}$$

where $q=\exp(-\pi K'/K)$, K is the complete elliptic integral to modulus k, and K' is the complementary integral. The dimensions of the conductors are given by

$$b/2 = K\left[\frac{1+k}{2k} - \frac{\pi}{2kK}\right],$$

$$g = K'\left[\frac{1+k}{2k}\right]; \qquad h = \frac{\pi}{2k}.$$

The correction length for one corner is the essential factor that enters into the determination of both the capacitance and inductance of the line. Fig. 8 gives the values of $C_f(g,b)/\epsilon = X/g$ as a function of b/g for the case $g=h$. It is to be noted that when the two sides of the inner conductor are short, the fringing capacitance for one corner is dependent on the short side b as well as on the spacings g and h. The case that b/g tends to infinity corresponds to a very thick inner conductor; then the fringing capacitance for one corner becomes twice that given in (6), and the correction length in (16) simplifies to

$$X = g\left[1 - \frac{2}{\pi} \log 2\right]. \tag{17}$$

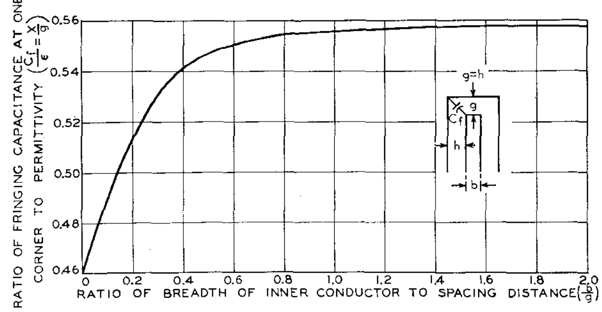

Fig. 8—Radio of fringing capacitance to permittivity (C_f/ϵ) as a function of the ratio of breadth of inner conductor to spacing distance (b/g).

The capacitance between the two conductors per unit length of line is

$$C = 2\epsilon\left(\frac{w+b}{g}\right) + 4C_f(g, b), \tag{18}$$

and the inductance of the line is determined from the equation

$$L = \frac{\mu g}{2(b+w)+4X}, \tag{19}$$

in which X is found from (16) or from Fig. 8. The characteristic impedance of the line is as follows:

$$Z = \frac{376.62}{2\left(\frac{w+b}{g}\right) + \frac{4C_f(g,h)}{\epsilon}} \text{ ohms.} \tag{20}$$

For the line having the dimensions $w=0.218$, $b=0.050$, and $g=h=0.050$ inch, the fringing capacitance found from Fig. 8 is $4C_f(g,h)=4(0.5537\epsilon)=2.215\epsilon$. The result calculated previously from the thick conductor formula is 2.232ϵ, which is very close to the above exact value.

[5] J. D. Cockcroft, "The effect of curved boundaries on the distribution of electrical stress round conductors," *J. IEE*, vol. 66, pp. 385–409; April, 1928.

V. Approximate Formulas for Thin Inner Conductors

The expressions for the determination of the capacitance of the rectangular lines having thin inner conductors and unequal spacings contain complex combinations of elliptic functions, and are very difficult to evaluate in applications. It becomes necessary to derive approximate formulas yielding capacitance and inductance values of an accuracy comparable to those obtained from Cockcroft's exact solutions.

A. Inner Conductor of Zero Thickness

The capacitance between the conductors in Fig. 9(a) can be divided into three parts: 1) capacitance of parallel-plate capacitors formed by the horizontal walls of the conductors, 2) capacitance due to fringing flux lines from each end of the inner conductor to the opposite horizontal walls of the outer one, and 3) fringing capacitance caused by the presence of the lateral walls of the outer conductor.

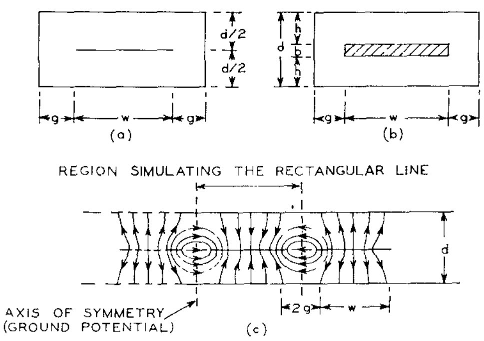

Fig. 9—(a) Rectangular line with inner conductor of zero thickness, (b) rectangular line with inner conductor of finite thickness, and (c) approximate field distribution in a shielded coupled-strip line operating in the odd mode.

The first capacitance can be readily determined. Inspection of Fig. 9(c) indicates that the disturbance of flux lines close to the ends of the inner conductor is not pronounced; consequently, the second item of capacitance can be neglected. The effect of sidewalls present in the outer conductor is the same as that obtained on the addition of a series of plates similar to the inner conductor spaced at $2g$ between their ends, as shown in Fig. 9(c). The dotted lines are at the same potential as the two infinite plates and can be taken to represent the sidewalls of the rectangular line. From the character of the flux distribution, it can be seen that because of the sidewalls, the fringing capacitance equals that between two coupled, odd-mode strips located midway between two infinite parallel plates. Cohn[6] gives the capacitance as follows:

$$C_{f0}\left(0, \frac{2g}{d}\right) = \frac{2\epsilon}{\pi} \log\left(1 + \coth\frac{\pi g}{d}\right), \qquad (21)$$

where d is the spacing between the infinite plates. The line capacitance is given by

$$C = \frac{4\epsilon w}{d} + 4\epsilon\left[\frac{2}{\pi}\log\left(1 + \coth\frac{\pi g}{d}\right)\right], \qquad (22)$$

and the characteristic impedance of the line is

$$Z = \frac{376.62}{4\left[\frac{w}{d} + \frac{2}{\pi}\log\left(1 + \coth\frac{\pi g}{d}\right)\right]} \text{ ohms.} \qquad (23)$$

If the dimensions of a rectangular line are $w = 0.300$, $d = 0.200$, $b = 0$, and $g = h = 0.100$ inch, the capacitance as found from Fig. 8 becomes $6\epsilon + 4(0.46)\epsilon = 7.84\epsilon$, whereas (22) gives $C = 6\epsilon + 1.88\epsilon = 7.88\epsilon$.

B. Inner Conductor of Finite Thickness

When the thickness of the inner conductor is appreciable, but is less than $d/3$, the line capacitance can be determined by the method adopted for the case of the infinitely thin inner conductor. However, the fringing capacitance caused by the sidewalls should be corrected for the finite thickness of the inner conductor in Fig. 9(b).

Consider a semi-infinite plate of thickness b located in the middle of two infinite parallel plates which are separated at a distance d, as shown by Fig. 15(a) in Appendix III. The semi-infinite plate is charged to a potential V, and the two infinite plates are at ground potential. The fringing capacitance for one corner of the semi-infinite plate as obtained by Thomson is[8]

$$C_f = \left(\frac{b}{d}\right) = \frac{\epsilon}{\pi}\left[\frac{d}{d-b}\log\frac{2d-b}{b} + \log\frac{b(2d-b)}{(d-b)^2}\right]. \qquad (24)$$

If the thickness of the middle plate is reduced to zero, the fringing capacitance at one corner is then expressed by

$$C_f(0) = \frac{2\epsilon}{\pi}\log 2 = 0.4407\epsilon. \qquad (25)$$

When applied to the rectangular line, the fringing capacitance for each corner of the inner conductor can be approximated by

$$C_f\left(\frac{b}{d}, \frac{2g}{d}\right) = C_{f0}\left(0, \frac{2g}{d}\right)\frac{C_f(b/d)}{C_f(0)}. \qquad (26)$$

[6] S. B. Cohn, "Shielded coupled-strip transmission line," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 29–38; October, 1955.

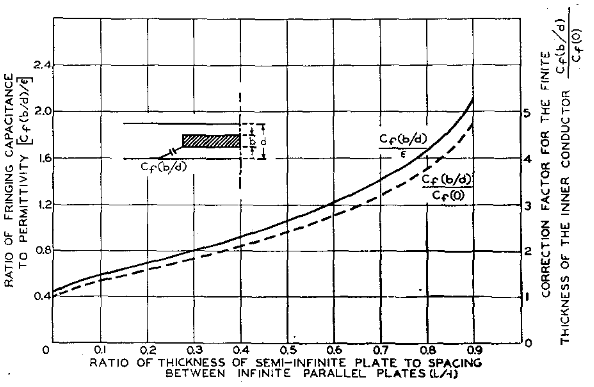

Fig. 10—Fringing capacitance at one corner for shielded semi-infinite plate and the correction factor for the thickness of the semi-infinite plate.

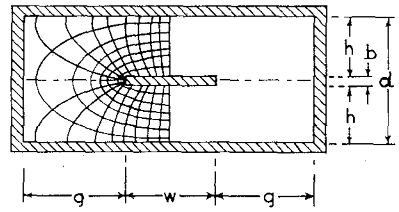

Fig. 11—Electric and magnetic fields in a rectangular line with narrow inner conductor.

The term $C_f(b/d)/\epsilon$ along with the ratio $C_f(b/d)/C_f(0)$ is plotted in Fig. 10 as a function of b/d. The complete expressions for the line capacitance and characteristic impedance are, respectively

$$C = \frac{4\epsilon w}{d-b} + 4\frac{\epsilon}{\pi}\left[\frac{d}{d-b}\log\frac{2d-b}{b} + \log\frac{b(2d-b)}{(d-b)^2}\right]\frac{\log\left(1+\coth\frac{\pi g}{d}\right)}{\log 2}, \quad (27)$$

$$Z = \frac{1}{4\epsilon\left\{\frac{w}{d-b} + \frac{1}{\pi}\left[\frac{d}{d-b}\log\frac{2d-b}{b} + \log\frac{b(2d-b)}{(d-b)^2}\right]\right\}v} \text{ ohms.} \quad (30)$$

and

$$Z = \frac{376.62}{4\left[\frac{w}{d-b} + \frac{1}{\epsilon}C_{f0}\left(0,\frac{2g}{d}\right)\frac{C_f(b/d)}{C_f(0)}\right]} \text{ ohms.} \quad (28)$$

A rectangular line has the description, $w=0.218$, $b=0.050$, $d=0.150$, and $g=h=0.050$ inch. Eqs. (21), (24)–(26) yield, respectively, $C_{f0}(0, 2g/d)=0.526\epsilon$, $C_f(b/d)=0.839\epsilon$, $C_f(0)=0.4407\epsilon$, and $C_f(b/d, 2g/d)=1.002\epsilon$. The total capacitance of the line is $C=8.72\epsilon+4(1.002\epsilon)=12.728\epsilon$, compared with the previously obtained results of 12.952ϵ and 12.935ϵ farads per meter.

A second line has the dimensions $w=0.180$, $b=0.030$, $g=h=0.060$ inch. The use of Fig. 8 gives the result:

$$C = 2(0.180+0.030)\epsilon/0.060 + 4(0.548)\epsilon = 9.192\epsilon.$$

The preceding formulas give $C_{f0}(0, 2g/d)=0.4954\epsilon$, $C_f(b/d)=0.694\epsilon$, $C_f(0)=0.4407\epsilon$, and $C_f(b/d, 2g/d)=0.781\epsilon$. These values give a total capacitance:

$$C = 2\epsilon(0.180)/0.06 + 4(0.781)\epsilon = 9.124\epsilon.$$

VI. Rectangular Lines with Narrow Inner Conductors

When the flat inner conductor is not more than one-quarter as wide as the outer conductor, the effect of the presence of sidewalls may be ignored.[7] Thus, in Fig. 11, the fringing field at each end of the inner conductor has the same character as that occurring in a semi-infinite plate situated between two infinite parallel plates. If the rectangular line has a narrow inner conductor of negligible thickness, the expression for characteristic impedance, when (25) is used, should be the following:

$$Z = \frac{1}{4\epsilon\left(\frac{w}{d} + \frac{2}{\pi}\log 2\right)v} \text{ ohms.} \quad (29)$$

When the inner conductor of the line has finite thickness, then by using (24),

The capacitance and inductance of such lines can be readily evaluated.

VII. Inner Conductor with Rounded Corners

In applications involving high voltages, the annular space between the conductors may be subjected to high electrical stress. The stress at the sharp corners of the inner conductor can reach as much as six times the mean value.[5] To alleviate this stress concentration, the edges of the inner conductor are rounded. Cockcroft has treated the case when the spacings g and h are equal and are small compared with conductor sides, and has found that the electrical stress is considerably reduced by rounding the edges, but the change in capacitance is insignificant. The fringing capacitance at one corner is plotted in Fig. 12 in the form C_f/ϵ, which is equivalent to X/g. The fringing effect decreases with the increase in the ratio of the radius of curvature R to the spacing g. The process of conformal representation for this case is given in Appendix IV.

[7] R. H. T. Bates, "The characteristic impedance of the shielded slab line," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 28–33; January, 1956.

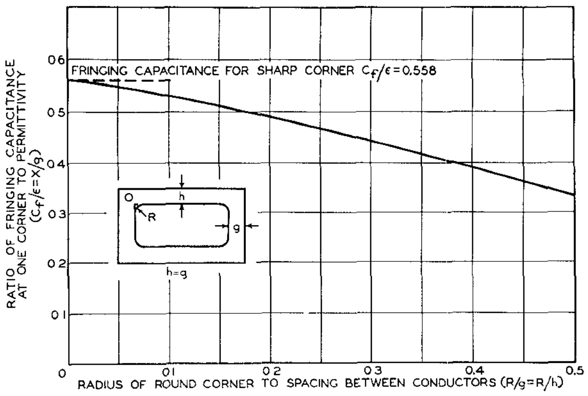

Fig. 12—Fringing capacitance for one round corner to permittivity as a function of the ratio of the corner radius to conductor spacing.

The fictitious increase in length of the sides may be utilized to calculate the capacitance and inductance of the line as follows:

$$C = 2\epsilon(w + b + 2X)/g, \tag{31}$$

and

$$L = \mu g/(2w + 2b + 4X). \tag{32}$$

The characteristic impedance can next be calculated from (2).

Appendix I

L-Shaped Bend

When the sides of the conductors are large compared with the spacings between them, one quarter of the cross section is to be considered. This portion of the cross section then constitutes a right-angled bend as shown in Fig. 13(a) and forms the polygon in the z-plane bounded by the lines ABC and DEF, which are maintained at potential V and zero, respectively. It can be assumed that $t = -\infty$ at a distant point on AB, $t = -a$ at B, $t = 0$ at a distant point on BC, and $t = 1$ at E. The internal angles of the polygon are $\pi/2$ at B, zero at C, and $3\pi/2$ at E. The quantity a has to be determined from the geometry of the system. The Schwarz-Christoffel transformation which turns the boundary of the polygon into the real axis in the t plane, shown in Fig. 13(b), is

$$\frac{dz}{dt} = A_1(t-1)^{1/2}(t-0)^{-1}(t-a)^{-1/2}. \tag{33}$$

The diagram in the w plane, shown in Fig. 13(c), consists of the real axis and a line parallel to it. The internal angle of the polygon is at $t = 0$, and is equal to zero. The transformation which turns this diagram into the real axis of the t plane is

$$\frac{dw}{dt} = B_1(t-1)^0(t-0)^{-1}(t-a)^0, \tag{34}$$

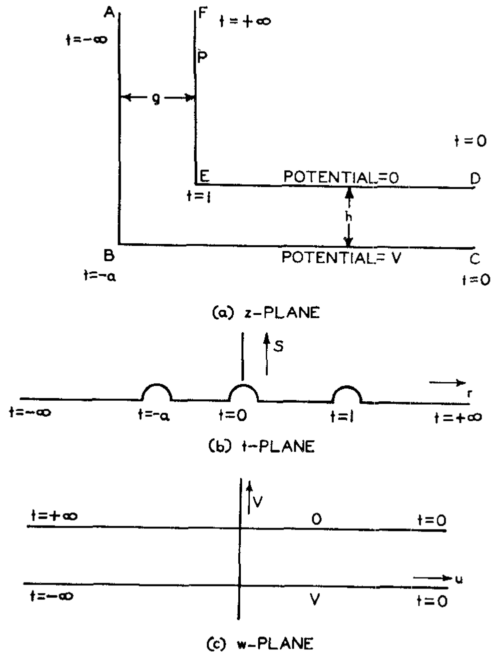

Fig. 13—Conformal transformation of the L-shaped bend.

which, upon integration and use of the boundary conditions, becomes

$$w = u + iv = \frac{V}{\pi}\log t. \tag{35}$$

To integrate (33), let

$$X = (t-1)^{1/2}(t-a)^{-1/2}, \tag{36}$$

and obtain

$$z = -2A_1(1/a)^{1/2}\arctan(a^{1/2}X) + A_1\log\frac{1+X}{1-X}, \tag{37}$$

in which point E is chosen as the origin in the z plane. Consideration of the values of t and X at point B leads to the relations

$$g = A_1\pi, \quad h = A_1\pi(1/a)^{1/2}, \quad (h/g)^2 = 1/a. \tag{38}$$

Let P be a remote point on the line EF. The total charge on the strip EP per unit length in the direction normal to the plane of the paper is given by [3,4]

$$Q_{EP} = \left(\frac{V}{\pi}\right)\epsilon\left[\log(a+1) + 2(1/a)^{1/2}\arctan a^{1/2} - 2\log 2 + \frac{\overline{EP}}{A_1}\right].$$

If the flux lines were not disturbed, the charge on the strip EP would be $(\overline{EP})\epsilon V/g$, which is the last term in the preceding expression. The first three terms represent the excess charge caused by the flux disturbances. When the excess charge is divided by V and the relations in (38) are used, the fringing capacitance C_{f1} is obtained as

$$C_{f1} = \frac{\epsilon}{\pi}\left[\log\frac{g^2+h^2}{4h^2} + 2\left(\frac{h}{g}\right)\arctan\frac{g}{h}\right]. \tag{39}$$

Similarly, the fringing capacitance associated with the horizontal side C_{f2} may be obtained from (39) by interchanging g and h.

Appendix II

Rectangular Line Having Thin Inner Conductor

If the two sides of the rectangular line are short compared with the conductor spacings, half of the cross section shown in Fig. 14 should be considered. The appropriate transformations are[5]

$$z = \int \frac{(1 - t^2)^{1/2}}{(1 - k^2t^2)^{1/2}} \frac{dt}{(1 - k^2 \operatorname{sn} \alpha \cdot t^2)}, \quad (40)$$

and

$$w = u + jv = \frac{1}{\pi} \log \frac{\left(\frac{1}{k \operatorname{sn} \alpha} - t\right)}{\left(\frac{1}{k \operatorname{sn} \alpha} + t\right)}. \quad (41)$$

Here, k is the modulus of elliptic functions, $\operatorname{sn} \alpha$ is one of the Jacobian elliptic functions, and α is a complex quantity.

These transformations yield the line capacitance as

$$C = 4(\overline{AC} + \overline{CD} + X)/g \text{ farad per meter,} \quad (42)$$

in which X is the correction for the sides at one corner resulting from the field distortion and is given by an expression comprising elliptic and other higher functions.

For the case $\alpha = K - jK'$, and for values of k from 0.5 to unity, the equation for X reduces to (16), and g does not differ substantially from h. The fringing capacitance at one corner has been plotted in Fig. 8 as a function of b/g in the form

$$C_f/\epsilon = X/g. \quad (43)$$

Appendix III

Shielded Semi-Infinite Plate

Fig. 15(a) shows a semi-infinite plate of finite thickness and rectangular cross section placed midway between two infinite plates. The two parallel infinite plates are at zero potential, and the semi-infinite plate is at potential V. The values of t chosen at the corners of the z-plane polygon are indicated in Fig. 15, both (a) and (b). The internal angles of the polygon are 0 when $t = \pm 1$, and $3\pi/2$ when $t = \pm a$. The differential equation which transforms the boundary of the z diagram into the real axis of the t plane is[3]

$$\frac{dz}{dt} = A_1(t + a)^{1/2}(t - a)^{1/2}(t + 1)^{-1}(t - 1)^{-1}. \quad (44)$$

The diagram in the w plane consists of one straight line and the two sides of another parallel line, as shown in Fig. 15(c). The internal angles occur at points corresponding to $t = -1$ and $t = 1$ and are both zero. The transformation which turns the diagram in the w plane

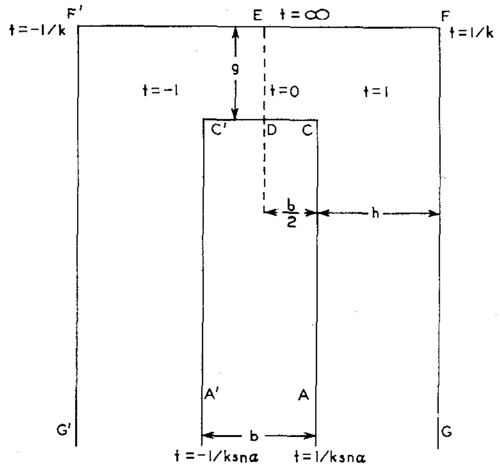

Fig. 14—Transformation for half cross section of a rectangular line.

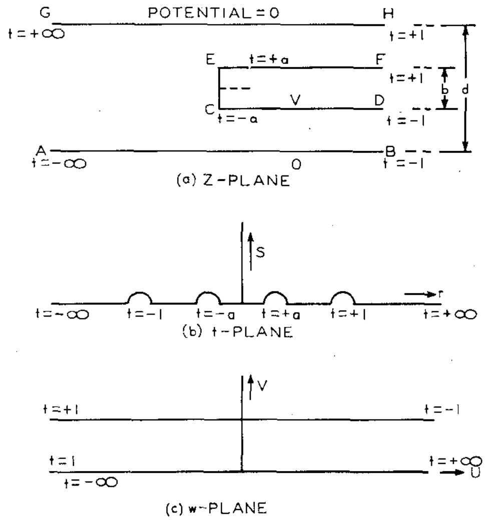

Fig. 15—Schwarz-Christoffel transformation for shielded semi-infinite plate.

to the real axis in the t plane is

$$\frac{dw}{dt} = B_1(t + 1)^{-1}(t - 1)^{-1}. \quad (45)$$

After (44) and (45) are integrated and the integration constants are determined by application of the boundary conditions, the fringing capacitance for one corner of the semi-infinite plate is given by (24).

Appendix IV

Inner Conductor with Rounded Edges

The case considered is that of two coaxial rectangular conductors, the edges of the inner conductor being rounded. A potential difference π is maintained between the conductors, each of which forms an equipotential surface. The distances g and h are assumed to be small compared with the sides AB and CD of Fig. 16(a), so that the z-plane polygon consists of one corner, the sides of which are assumed to extend to infinity. The

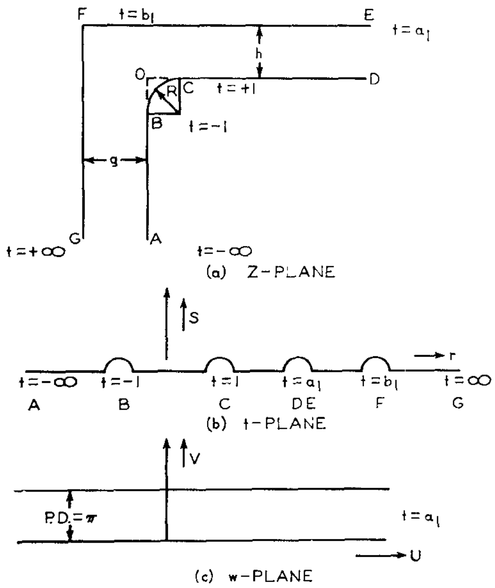

Fig. 16—Conformal transformation of a rounded bend.

differential equation which transforms the polygon $ABCDEGA$ in the z plane to the upper half of the t plane is[5]

$$\frac{dz}{dt} = A_1 \frac{(t+1) + m(t-1)}{(t-a_1)(t-b_1)^{1/2}} \cdot \tag{46}$$

The two terms in the numerator have the effect of turning the path of z through 90° as r in the t plane of Fig. 16(b) varies between -1 and $+1$. If the factor m is chosen as

$$m = (b_1+1)^{1/2}/(b_1-1)^{1/2}, \tag{47}$$

then the variation of the electric field intensity is uniform along the curved boundary BC.

It is necessary to put an electric field into the t plane such that from $t=-\infty$ to $t=a_1$, the horizontal line is at one potential and the horizontal line from $t=a_1$ to $t=+\infty$ at another potential. This is accomplished by opening up the two parallel lines in Fig. 16(c) to form the two lines AD and EG in the t plane. The transformation is

$$\frac{dw}{dt} = \frac{B_1}{t-a_1} \cdot \tag{48}$$

After integration and evaluation of the total charge on the inner conductor as z takes infinite real and imaginary values; that is, as t varies from $-\infty$ to $+a_1$, the total charge per unit length is given by

$$Q = \epsilon\pi\left[\frac{(OA+X_1)}{g} + \frac{(OD+X_2)}{h}\right], \tag{49}$$

where OA and OD are, respectively, half the depth and half the width of the inner conductor. The expressions for the correction lengths X_1 and X_2 have been derived by Cockcroft, and the excess or fringing capacitance for one rounded corner is found from

$$C_f = \frac{Q}{\pi} = \epsilon\left[\left(\frac{X_1}{g} + \frac{X_2}{h}\right)\right]. \tag{50}$$

The most usual case is for $g=h$. Under this condition,

$$\frac{b_1-a_1}{a_1-1} = \frac{b_1+1}{b_1-1} = \frac{a_1+1}{b_1-a_1} = m^2, \tag{51}$$

and

$$g = h = \pi(1+m). \tag{52}$$

The radius of the round corner is given by

$$R = 2 \arctan i - m \log \frac{1+f}{1-f}, \tag{53}$$

in which the expressions for f and i are, respectively,

$$f = \left(\frac{2}{b_1+1}\right)^{1/2} \quad \text{and} \quad i = \left(\frac{2}{b_1-1}\right)^{1/2}.$$

The expression for the total correction length and the fringing capacitance for one rounded corner reduce to

$$X = 2\pi + 4(m-1)\arctan\left(\frac{1}{m}\right) - 2(1+m)\log 4 + 2(1+m)\log 2b_1 - 2m\log(b_1+1) - 2\log(b_1-1), \tag{54}$$

and

$$C_f = \frac{\epsilon X}{g} \cdot \tag{55}$$

The values of $C_f/\epsilon = X/g$ are plotted in Fig. 12 as a function of R/g. In general, when g is not equal to h, the results are far more complex than those given by the previous equations and the curved boundary at the corner of the inner conductor does not assume a circular form.

VIII. Acknowledgment

Acknowledgment is due W. N. Parker, Electron Tube Division, RCA, Lancaster, Pa., for very helpful discussions and for providing his results obtained by relaxation method on rectangular lines.